

Commensurate anisotropic oscillator, $SU(2)$ coherent states and the classical limit

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Abstract. We demonstrate a formally exact quantum-classical correspondence between the stationary coherent states associated with the commensurate anisotropic two-dimensional harmonic oscillator and the classical Lissajous orbits. Our derivation draws upon earlier work of Louck *et al* [1973 *J. Math. Phys.* **14** 692] wherein they have provided a non-bijective canonical transformation that maps, within a degenerate eigenspace, the commensurate anisotropic oscillator on to the isotropic oscillator. This mapping leads, in a natural manner, to a Schwinger realization of $SU(2)$ in terms of the canonically transformed creation and annihilation operators. Through the corresponding coherent states built over a degenerate eigenspace, we directly effect the classical limit via the expectation values of the underlying generators. Our work completely accounts for the fact that the $SU(2)$ coherent state in general corresponds to an ensemble of Lissajous orbits.

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1. Introduction

The anisotropic oscillator has long been of relevance in describing the intrinsic states of a deformed nucleus in the Nilsson Model [1]. The discovery of super-deformed high spin states of some nuclei [2] corresponding to spheroidal nuclear shapes of approximately commensurate axial lengths had helped focus attention on the commensurate anisotropic oscillator. Similarly in quantum optics a two-mode radiation field may also be discussed in terms of a two-dimensional oscillator [3]. Likewise in condensed matter physics, the design of nanostructures permitting ballistic motion of electrons [4, 5] represents yet another area for the application of such studies.

Considerable attention has been paid in the literature on the question of symmetries and degeneracies in the commensurate anisotropic oscillator [6, 7, 8, 9, 10]. In particular Louck *et al* have addressed this question from a group theoretical viewpoint by studying the non-bijective canonical transformation that maps the commensurate anisotropic oscillator, within a degenerate eigenspace, to the isotropic one.

While the question of achieving the classical limit of quantum dynamics of simple systems via appropriately constructed coherent states [11, 12, 13, 14] has been a long-standing one, interesting experiments have been carried out recently to demonstrate such a classical limit in quantum systems [15]. More recently the classical limit of the commensurate anisotropic oscillator has been investigated experimentally [16] in a laser resonator by exploiting the analogy between the Schrödinger equation for the two-dimensional harmonic oscillator and the paraxial wave equation for the spherical resonators [17, 18]. The question of analytically demonstrating the classical limit in this system via appropriately constructed coherent states and accounting for the experimentally observed wave patterns has been an intriguing one and has been addressed by various authors [16, 19, 20, 21, 22]. The purpose of this paper is to resolve this question using an approach that exploits the symmetry properties of the commensurate two-dimensional anisotropic oscillator well studied in the literature [8].

Consider the two-dimensional harmonic oscillator described by the Hamiltonian [21],

$$H = \frac{1}{2} \left(\hat{p}_x^2 + \hat{p}_y^2 + \omega_1^2 \hat{x}^2 + \omega_2^2 \hat{y}^2 \right), \quad (1)$$

where $\omega_1 = q\omega$ and $\omega_2 = p\omega$, ω is the common factor of the frequencies ω_1 and ω_2 , and p and q are integers. Normally one takes p and q to be coprime, without loss of generality, as the common factor between p and q (M say) can be absorbed in the definition of the common frequency ω . However, we take the Hamiltonian (1) here to describe the experimental situation of Chen *et al* [16] where the common frequency ω represents the transverse mode spacing in the spherical resonator, and p and q can be independently varied by suitably tuning the cavity length and appropriately choosing the longitudinal mode indices. Thus p and q could in practice have a common factor $M \neq 1$. Further, as reported by Chen *et al* [16], the experimental situations corresponding to the choice of parameters (p, q) and (lp, lq) where l is a positive integer, give rise to qualitatively different results in regard to quantum-classical correspondence. In view of this, in the rest of the paper, we take p and q to be having a common factor M in general.

The Hamiltonian (1) can be written in terms of the creation and annihilation operators in the form,

$$H = \omega' \left[\frac{1}{p} \left(a_1^\dagger a_1 + \frac{1}{2} \right) + \frac{1}{q} \left(a_2^\dagger a_2 + \frac{1}{2} \right) \right], \quad (2)$$

where $\omega' = \omega pq$ and

$$a_1 = \frac{1}{\sqrt{2q\omega}} (q\omega \hat{x} + i\hat{p}_x), \quad a_2 = \frac{1}{\sqrt{2p\omega}} (p\omega \hat{y} + i\hat{p}_y). \quad (3)$$

It is in fact straightforward to achieve the classical limit of the quantum dynamics described by the Hamiltonian (2) via the two-mode harmonic oscillator coherent states $|\alpha_1, \alpha_2\rangle$ that are defined by

$$a_1 |\alpha_1, \alpha_2\rangle = \alpha_1 |\alpha_1, \alpha_2\rangle, \quad a_2 |\alpha_1, \alpha_2\rangle = \alpha_2 |\alpha_1, \alpha_2\rangle. \quad (4)$$

Note that these are the coherent states associated with the Heisenberg-Weyl group [24]. Let the system be initially (at $t = 0$) in the two-mode coherent state $|\alpha_1, \alpha_2\rangle$. The expectation values of a_1, a_2 in this state evolve in time under the Hamiltonian (2) as

$$\langle a_1(t) \rangle = \alpha_1 e^{-iq\omega t}, \quad \langle a_2(t) \rangle = \alpha_2 e^{-ip\omega t}. \quad (5)$$

The classical Hamiltonian corresponding to (1) can be rewritten in the form

$$H = \omega' \left(\frac{1}{p} |z_1|^2 + \frac{1}{q} |z_2|^2 \right), \quad (6)$$

where the complex variables (z_1, z_2) are related to the classical coordinates x, y and momenta p_x, p_y by

$$z_1 = \frac{1}{\sqrt{2q\omega}} (q\omega x + ip_x), \quad z_2 = \frac{1}{\sqrt{2p\omega}} (p\omega y + ip_y), \quad (7)$$

with $\omega' = \omega pq$ as defined earlier. Let us write the solutions of the classical Hamiltonian (6) as

$$z_1(t) = \sqrt{\frac{\omega q}{2}} \eta_1 e^{-i(\omega q t - \phi_1)}, \quad z_2(t) = \sqrt{\frac{\omega p}{2}} \eta_2 e^{-i(\omega p t - \phi_2)}, \quad (8)$$

so that the equations describing the classical Lissajous orbits would be given by

$$x(t) = \eta_1 \cos(q\omega t - \phi_1), \quad y(t) = \eta_2 \cos(p\omega t - \phi_2). \quad (9)$$

The position probability density, namely, $|\langle x, y | \alpha_1, \alpha_2 \rangle|^2$ is Gaussian centred at $[(\langle a_1 \rangle + \langle a_1^\dagger \rangle) / \sqrt{2q\omega}, (\langle a_2 \rangle + \langle a_2^\dagger \rangle) / \sqrt{2p\omega}]$, and becomes localized at this point in the classical limit, *i.e.*, $\hbar \rightarrow 0$. Thus as time evolves the peak of the position probability density, rides on the classical trajectory (9). This suggests the following prescription for implementing the classical limit: the expectation values of the generators $a_1, a_1^\dagger, a_2, a_2^\dagger$, of the Heisenberg-Weyl group, in the two-mode coherent state, tend to the corresponding classical values. Thus the classical limit in this case is obtained simply by making the correspondence

$$(\langle a_1 \rangle, \langle a_1^\dagger \rangle, \langle a_2 \rangle, \langle a_2^\dagger \rangle) \longrightarrow (z_1, z_1^*, z_2, z_2^*). \quad (10)$$

The above correspondence is also evident from the formal similarity between the solutions (5) for the expectation values and the solutions (8) for the corresponding classical dynamical variables. This correspondence yields a relation between the parameters in the equations for the Lissajous orbits (9) and the coherent state $|\alpha_1, \alpha_2\rangle$ as

$$\alpha_1 = \sqrt{\frac{\omega q}{2}} \eta_1 e^{i\phi_1}, \quad \alpha_2 = \sqrt{\frac{\omega p}{2}} \eta_2 e^{i\phi_2}. \quad (11)$$

Thus there is a unique classical trajectory corresponding to a given two-mode coherent state.

Note that the demonstration of the classical limit of the two-dimensional oscillator that we have presented above, via the two-mode coherent state $|\alpha_1, \alpha_2\rangle$, would be valid even if one considers the two frequencies ω_1, ω_2 to be incommensurate. This in fact is an unsatisfactory feature since the coherent state $|\alpha_1, \alpha_2\rangle$ does not embody the full

symmetry of the commensurate anisotropic oscillator Hamiltonian. To illustrate this point, let us look at the special case of the isotropic oscillator ($p = q = 1$). The $SU(2)$ symmetry in this case is manifest as the classical Hamiltonian (6) preserves the form $|z_1|^2 + |z_2|^2$. The quantum Hamiltonian (2) on the other hand can be rewritten as

$$H = \omega'(2J_0 + 1), \quad (12)$$

where J_0 is the Casimir operator corresponding to the $SU(2)$ Lie algebra generated, in the Schwinger realization, by

$$J_+ = a_1^\dagger a_2, \quad J_- = a_1 a_2^\dagger, \quad J_z = (a_1^\dagger a_1 - a_2^\dagger a_2)/2, \quad J_0 = (a_1^\dagger a_1 + a_2^\dagger a_2)/2. \quad (13)$$

Here the operators J_\pm , J_z obey the standard commutation relations

$$[J_z, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_z. \quad (14)$$

In view of (13), the set of simultaneous eigenstates of $J^2 = J_0(J_0 + 1)$ and J_z , namely $|j, m\rangle$, $j = 0, 1, \dots, \infty$, $|m| \leq j$, where $j = \frac{1}{2}(n_1 + n_2)$ and $m = \frac{1}{2}(n_1 - n_2)$ is isomorphic to the set of number states $|n_1, n_2\rangle$. The isotropic oscillator Hamiltonian divides this set of states into degenerate eigenspaces each characterized j (the eigenvalue of the Casimir operator J_0) independent of m . The two-mode harmonic oscillator coherent state $|\alpha_1, \alpha_2\rangle$ can then be expressed as [25]

$$|\alpha_1, \alpha_2\rangle = \sum_{j=0}^{\infty} e^{-\frac{1}{2}(|\alpha_1|^2 + |\alpha_2|^2)} (|\alpha_1|^2 + |\alpha_2|^2)^j \left(\frac{\alpha_2}{|\alpha_2|} \right)^j |j, \tau\rangle, \quad (15)$$

where $|j, \tau\rangle$ is the $SU(2)$ coherent state [26, 27, 24] built over states in the degenerate eigenspace $\{|j, m\rangle, |m| \leq j\}$ (equivalently the number states $|n_1, n_2\rangle$ with $n_1 + n_2$ held fixed) namely,

$$|j, \tau\rangle = \frac{1}{(1 + |\tau|^2)^j} \sum_{m=-j}^j \binom{2j}{j+m}^{\frac{1}{2}} \tau^{j+m} |j, m\rangle, \quad (16)$$

with $\tau = \alpha_1/\alpha_2$. Note that the two-mode coherent state $|\alpha_1, \alpha_2\rangle$ involves a sum over all degenerate eigenspaces labelled by j , and hence it does not implement the $SU(2)$ symmetry of the isotropic oscillator Hamiltonian. The coherent state $|j, \tau\rangle$ on the other hand, being a projection of the two-mode coherent state on to a particular degenerate eigenspace characterized by the energy $E = \omega'(2j + 1)$, does respect this symmetry. In this sense the appropriate coherent state which must be used to examine the classical limit in the isotropic oscillator case is the $SU(2)$ coherent state (16).

Indeed, Bièvre [28] and Pollet *et al* [29] have used the $SU(2)$ coherent states to demonstrate the classical limit in the case of the isotropic harmonic oscillator. In particular they have rigorously demonstrated that the coordinate space probability density $|\langle x, y | j, \tau \rangle|^2$ in the limit $2j = N \rightarrow \infty$ becomes localized over the classical Lissajous (elliptic) orbits.

More recently, the question of how to analytically derive a connection between a suitably constructed coherent state for the commensurate two-dimensional anisotropic oscillator and the classical Lissajous orbits has acquired interest [19, 20, 21, 22],

especially with a view to theoretically account for the experimental demonstration of such a classical limit by Chen *et al* [16]. Chen and coworkers [16, 19] have made an ansatz on the appropriate coherent state, something that resembles an $SU(2)$ coherent state, namely,

$$|N, p, q, \tau\rangle = \frac{1}{(1 + |\tau|^2)^{N/2}} \sum_{K=0}^N \binom{N}{K}^{\frac{1}{2}} \tau^K |pK, q(N-K)\rangle, \quad (17)$$

where N is a non-negative integer. From a numerical study of the coordinate space probability density associated with the above state they have guessed the following quantum-classical connection. For coprime p and q the classical periodic orbit is given by

$$x(t) = \eta_1 \cos(q\omega t - \phi/p), \quad y(t) = \eta_2 \cos(p\omega t), \quad (18)$$

with the amplitudes η_1 and η_2 given by

$$\eta_1 = \sqrt{\frac{1}{\omega q} \left(\frac{2pN|\tau|^2}{1 + |\tau|^2} + 1 \right)}, \quad \eta_2 = \sqrt{\frac{1}{\omega p} \left(\frac{2qN}{1 + |\tau|^2} + 1 \right)}, \quad (19)$$

where ϕ is an arbitrary phase. On the other hand if p and q have a common factor M , the coordinate space probability density is found to correspond to an ensemble of classical periodic orbits, the total number of such periodic orbits being M , and their trajectories are given by

$$\begin{aligned} x_k(t) &= \eta_1 \cos[q\omega t - (\phi + 2\pi k)/p], & k = 0, 1, \dots, M-1, \\ y(t) &= \eta_2 \cos(p\omega t), \end{aligned} \quad (20)$$

with η_1, η_2 as defined in (19).

Chen *et al* [21] have, for the first time, attempted to give an analytical derivation of the above guessed equations for the classical periodic orbits in the commensurate anisotropic oscillator case. They effect the classical limit via the two-mode coherent state $|\alpha_1, \alpha_2\rangle$ as demonstrated above in equations (10) and (11), and then utilize the method of triangular partial sums to essentially project a stationary ‘coherent’ state out of the two-mode coherent state $|\alpha_1, \alpha_2\rangle$. They indicate a connection between the parameters of this stationary coherent state and the classical periodic orbits in the case when p and q are coprime, leaving the question of what happens in the case when p and q have a common factor $M \neq 1$ unanswered. Unfortunately, their derivation does not clearly bring out the fact that the coherent state that they have projected out is indeed the $SU(2)$ coherent state. Not surprisingly these authors have referred to the stationary state constructed by them as *a kind of* $SU(2)$ coherent state. Górska *et al* [22] on the other hand offer an approximate correspondence between the experimentally observed wave patterns and the classical Lissajous orbits.

It is natural to expect, as in the isotropic oscillator case, that the appropriate coherent states for the commensurate anisotropic oscillator should be those associated with its underlying symmetry group. While the underlying $SU(2)$ group structure of the isotropic oscillator is manifest, as outlined above, the fact that the group $SU(2)$

also captures the symmetry of the two-dimensional commensurate anisotropic oscillator has been shown by Louck *et al* [8]. In particular they have concentrated on the degenerate eigenspaces of the commensurate anisotropic oscillator and have constructed a non-bijective canonical transformation that maps, within a degenerate eigenspace, the commensurate anisotropic oscillator Hamiltonian to an isotropic one, thus revealing the $SU(2)$ symmetry and also accounting for the ‘accidental’ degeneracy in the former case. Furthermore, they have also noted that this mapping leads, in a natural manner, to a Schwinger realization of $SU(2)$ in terms of the canonically transformed creation and annihilation operators, within a given degenerate eigenspace.

In the present paper we use symmetry arguments to identify the appropriate coherent states for the commensurate anisotropic oscillator. We utilize the above-mentioned canonical transformation of Louck *et al* , and the Schwinger realization of $SU(2)$ to construct the stationary coherent states built over a degenerate subspace. We use these coherent states and demonstrate a correspondence with the classical Lissajous orbits. In particular we derive a relation between the parameters characterizing the $SU(2)$ coherent state and those characterizing the single Lissajous orbit in the case when p and q are coprime, and an ensemble of M Lissajous orbits when p and q have a common factor M .

2. Canonical transformations and the symmetry group of the commensurate anisotropic oscillator

In this section we collect the main results from the work of Louck *et al* [8] that we shall make use of in the next section. As has been shown by Louck *et al* [8], the eigenstates of the commensurate anisotropic oscillator Hamiltonian [with $\omega_1 = q\omega, \omega_2 = p\omega$] can be divided into qp number of different subsets of states [23]

$$\{|n_1p + \lambda_1, n_2q + \lambda_2\rangle, \quad n_1, n_2 = 0, 1, \dots, \infty\}, \quad (21)$$

for each $\lambda_1 = 0, 1, \dots, p-1, \lambda_2 = 0, 1, \dots, q-1$. The states in (21) are eigenstates of H with eigenvalues

$$E = \omega' \left[(n_1 + n_2) + \frac{1}{p} \left(\lambda_1 + \frac{1}{2} \right) + \frac{1}{q} \left(\lambda_2 + \frac{1}{2} \right) \right], \quad (22)$$

so that those states belonging to the set (21) for a fixed value of $n_1 + n_2$ are degenerate.

In each of the degenerate eigenspaces (21) labelled by (λ_1, λ_2) , there exists a canonical transformation $(a_1, a_2) \rightarrow (\tilde{a}_1, \tilde{a}_2)$ given by

$$\begin{aligned} \tilde{a}_1 &= \sqrt{\frac{1}{p}(\hat{n}_1 - \lambda_1)} \quad \hat{n}_1(\hat{n}_1 - 1)(\hat{n}_1 - p + 1)^{-\frac{1}{2}}(a_1^\dagger)^p, \\ \tilde{a}_2 &= \sqrt{\frac{1}{q}(\hat{n}_2 - \lambda_2)} \quad \hat{n}_2(\hat{n}_2 - 1)(\hat{n}_2 - q + 1)^{-\frac{1}{2}}(a_2^\dagger)^q, \\ \hat{n}_1 &= a_1^\dagger a_1, \quad \hat{n}_2 = a_2^\dagger a_2, \end{aligned} \quad (23)$$

such that the Hamiltonian in the transformed picture becomes that of an isotropic oscillator with frequency ω' , namely,

$$H = \omega' \left[\left(\tilde{a}_1^\dagger \tilde{a}_1 + \frac{1}{2} \right) + \left(\tilde{a}_2^\dagger \tilde{a}_2 + \frac{1}{2} \right) \right]. \quad (24)$$

Note that the action of the canonically transformed creation and annihilation operators on a particular state in the subset of states (21) is given by, for example,

$$\begin{aligned} \tilde{a}_1^\dagger |n_1 p + \lambda_1, n_2 q + \lambda_2\rangle &= \sqrt{n_1 + 1} |(n_1 + 1)p + \lambda_1, n_2 q + \lambda_2\rangle, \\ \tilde{a}_2 |n_1 p + \lambda_1, n_2 q + \lambda_2\rangle &= \sqrt{n_2} |n_1 p + \lambda_1, (n_2 - 1)q + \lambda_2\rangle, \end{aligned} \quad (25)$$

and so on.

As observed by Louck *et al* [8], one has the Schwinger realization of $SU(2)$ in terms of the canonically transformed operators $\tilde{a}_1, \tilde{a}_1^\dagger, \tilde{a}_2, \tilde{a}_2^\dagger$, namely,

$$J_+ = \tilde{a}_1^\dagger \tilde{a}_2, \quad J_- = \tilde{a}_1 \tilde{a}_2^\dagger, \quad J_z = (\tilde{a}_1^\dagger \tilde{a}_1 - \tilde{a}_2^\dagger \tilde{a}_2)/2, \quad J_0 = (\tilde{a}_1^\dagger \tilde{a}_1 + \tilde{a}_2^\dagger \tilde{a}_2)/2, \quad (26)$$

where the operators J_\pm, J_z obey the commutation relations (14).

In view of (25), one can identify, for fixed (λ_1, λ_2) , the simultaneous eigenstates of $J^2 = J_0(J_0 + 1)$ and J_z , namely $|j, m\rangle$, where $j = \frac{1}{2}(n_1 + n_2)$ and $m = \frac{1}{2}(n_1 - n_2)$, with $|n_1 p + \lambda_1, n_2 q + \lambda_2\rangle$. In terms of the generators of $SU(2)$ defined in (26) the Hamiltonian (24) is given by

$$H = \omega'(2J_0 + 1), \quad (27)$$

so that for fixed (λ_1, λ_2) , the energy eigenvalue in the state $|j, m\rangle$ is given by $E = \omega'(2j + 1)$ independent of m . This again reveals the ‘accidental’ degeneracy of the commensurate anisotropic oscillator due to the underlying $SU(2)$ symmetry group.

We would like to recall here that Louck *et al* [8] have also provided a canonical transformation $(z_1, z_2) \rightarrow (\tilde{z}_1, \tilde{z}_2)$, in terms of complex variables defined in (7), given by

$$\tilde{z}_1 = \frac{1}{\sqrt{p}} \left(\frac{z_1}{|z_1|} \right)^p |z_1|, \quad \tilde{z}_2 = \frac{1}{\sqrt{q}} \left(\frac{z_2}{|z_2|} \right)^q |z_2|, \quad (28)$$

such that the classical Hamiltonian (6) in the transformed picture becomes that of the classical isotropic harmonic oscillator, namely,

$$H = \omega' (|\tilde{z}_1|^2 + |\tilde{z}_2|^2). \quad (29)$$

The $SU(2)$ symmetry of the classical Hamiltonian (29) is evident again due to the fact that the form $|\tilde{z}_1|^2 + |\tilde{z}_2|^2$ is preserved.

3. $SU(2)$ coherent states, stereographic projection and Lissajous orbits

Let us construct a $SU(2)$ coherent state out of states in the degenerate eigenspace $\{|j, m\rangle, |m| \leq j\}$ for fixed (λ_1, λ_2) , namely,

$$\begin{aligned} |j, \tau\rangle &= \frac{1}{(1 + |\tau|^2)^j} \sum_{m=-j}^j \binom{2j}{j+m}^{\frac{1}{2}} \tau^{j+m} |j, m\rangle, \\ \tau &= \tan \frac{\theta}{2} e^{i\phi}, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi. \end{aligned} \quad (30)$$

We would like to remark that, in view of the isomorphism between the states $|n_1p + \lambda_1, n_2q + \lambda_2\rangle$ and the angular momentum eigenstates $|j, m\rangle$, for fixed (λ_1, λ_2) , where $j = \frac{1}{2}(n_1 + n_2)$ and $m = \frac{1}{2}(n_1 - n_2)$, one can see that the $SU(2)$ coherent state defined above is equivalent to the ‘coherent’ state (17) considered earlier by Chen and coworkers [16, 19, 21] if one makes the identification $N = 2j$ and specializes to $(\lambda_1, \lambda_2) = (0, 0)$. Thus we have provided a symmetry-based justification for the particular form of the coherent state, that Chen and coworkers had only conjectured based on heuristic considerations. As will become evident from the following analysis, the classical limit is independent of the choice of λ_1, λ_2 , *i.e.*, it does not matter which degenerate eigenspace one works in.

Let the system be initially (at $t = 0$) in the $SU(2)$ coherent state as defined in (30). As time evolves the system remains in the initial coherent state except for an irrelevant phase factor $e^{-i\omega't(2j+1)}$, so that the expectation values of $J_x = (J_+ + J_-)/2$, $J_y = -i(J_+ - J_-)/2$, and J_z remain stationary and are given by

$$\langle J_x \rangle = j \sin \theta \cos \phi, \quad \langle J_y \rangle = -j \sin \theta \sin \phi, \quad \langle J_z \rangle = -j \cos \theta. \quad (31)$$

Clearly the point $(\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle)$ lies on a sphere of radius j .

Let us next consider the solutions of the classical Hamiltonian (6) given in (8). In terms of the canonically transformed complex variables $(\tilde{z}_1, \tilde{z}_2)$ defined in (28) these solutions become

$$\tilde{z}_1(t) = \frac{1}{\sqrt{p}} \sqrt{\frac{\omega q}{2}} \eta_1 e^{-ip(\omega q t - \phi_1)}, \quad \tilde{z}_2(t) = \frac{1}{\sqrt{q}} \sqrt{\frac{\omega p}{2}} \eta_2 e^{-iq(\omega p t - \phi_2)}. \quad (32)$$

Note that while the solutions generated by the classical Hamiltonian (6), namely $z_1(t)$ and $z_2(t)$, oscillate at frequencies ωp and ωq respectively, the solutions generated by the canonically transformed Hamiltonian (20) oscillate at the common frequency $\omega p q$. We would like to remark that the canonical transformation (28) given by Louck *et al*, although it is a transformation of phase space variables, when regarded as a transformation among the coordinates alone, amounts to an *untwisting* of the Lissajous figures into a generic ellipse.

Since $|\tilde{z}_1|^2 + |\tilde{z}_2|^2$ is a constant in view of energy conservation (29), there exists a mapping (stereographic projection) from a point (j_x, j_y, j_z) on a sphere of radius j , via the north pole, to the complex Z -plane where we have defined Z to be

$$Z = 2j \frac{\tilde{z}_2}{\tilde{z}_1}. \quad (33)$$

The stereographic projection from (j_x, j_y, j_z) to Z is given by

$$Z = \frac{2j}{j - j_z} (j_x + i j_y). \quad (34)$$

Recall that we have earlier effected the transition to the classical limit (10) of the two-dimensional oscillator by identifying the expectation values of the generators of the Heisenberg-Weyl group, namely $a_1, a_1^\dagger, a_2, a_2^\dagger$, in the two-mode coherent states, with the corresponding classical phase space values. Motivated by this we prescribe that the transition to the classical limit of the commensurate anisotropic two-dimensional

oscillator, in terms of the $SU(2)$ coherent state, can be effected in analogy with (10), by making the correspondence between the expectation values of the generators of $SU(2)$ in the $SU(2)$ coherent states, namely,

$$(\langle J_x \rangle, \langle J_y \rangle, \langle J_z \rangle) \longrightarrow (j_x, j_y, j_z), \quad (35)$$

where (j_x, j_y, j_z) is the point on the sphere of radius j corresponding to the pair of complex numbers $(\tilde{z}_1, \tilde{z}_2)$ that form the solution set (32) of the classical isotropic oscillator Hamiltonian in the transformed picture (29). In fact such a quantum-classical correspondence is implicit in the analysis of Bièvre [28] and Pollet *et al* [29] in the case of the isotropic oscillator.

In view of the above proposed correspondence (35) we therefore have

$$j_x = j \sin \theta \cos \phi, \quad j_y = -j \sin \theta \sin \phi, \quad j_z = -j \cos \theta, \quad (36)$$

and hence in view of (34) the complex variable Z in the projective plane is related to the parameters in the $SU(2)$ coherent state (30) by

$$Z = 2j \cot \frac{\theta}{2} e^{-i\phi} = \frac{2j}{\tau}. \quad (37)$$

Upon combining this result with (32) and (33) we have the relations

$$\frac{q\eta_1}{p\eta_2} = |\tau| \quad (38)$$

and

$$e^{i(p\phi_1 - q\phi_2)} = e^{i\phi}. \quad (39)$$

The relation (38), in conjunction with the identification of the classical expression of energy (6) with the eigenvalue of the quantum Hamiltonian (27) in the $SU(2)$ coherent state, namely,

$$\omega^2 \left(\frac{q^2}{2} \eta_1^2 + \frac{p^2}{2} \eta_2^2 \right) = \omega'(2j+1) = \omega pq(N+1), \quad (40)$$

leads to the solutions for η_1 and η_2 ,

$$\eta_1 = \sqrt{\frac{2p(N+1)}{q\omega}} \frac{|\tau|}{\sqrt{1+|\tau|^2}}, \quad \eta_2 = \sqrt{\frac{2q(N+1)}{p\omega}} \frac{1}{\sqrt{1+|\tau|^2}}. \quad (41)$$

The solution of (39) needs detailed consideration. The general solution of the relation (39) may be written as

$$p\phi_1 - q\phi_2 = \phi + 2\pi k, \quad (42)$$

where k is an arbitrary integer. We shall now try to fix the allowed range of values of k . As we shall see this will depend on whether p and q are coprime or not. Note that keeping ϕ_1 fixed for example while varying ϕ_2 in equation (9) would only change the initial point on the Lissajous orbit and hence would leave the shape of the orbit itself invariant. This reparametrization invariance of the Lissajous orbits allows one

the freedom to choose ϕ_1 and ϕ_2 independently in such a way that (42) is valid. We conveniently choose

$$\phi_1 = \nu_1 \chi + \frac{\epsilon}{p}, \quad \phi_2 = -\nu_2 \chi + \frac{\epsilon}{q}, \quad (43)$$

where ν_1 and ν_2 are integers and ϵ is real. Hence the relation (42) now becomes

$$(p\nu_1 + q\nu_2)\chi = \phi + 2\pi k. \quad (44)$$

We now invoke the Bezout's identity [30] which states that there exist integers ν_1 and ν_2 such that one can always express the greatest common divisor of p and q (M say) in the form $p\nu_1 + q\nu_2 = M$. Hence it follows from Bezout's identity that if we choose ν_1 and ν_2 such that $p\nu_1 + q\nu_2 = M$ then the relation (44) becomes

$$\chi = \frac{\phi}{M} + \frac{2\pi k}{M}, \quad k = 0, 1, \dots, M-1. \quad (45)$$

If one assumes that p and q are relatively prime (*i.e.*, $M = 1$), then $k = 0$ is the only possibility in (42) and we thus get the unique solution

$$p\phi_1 - q\phi_2 = \phi. \quad (46)$$

On the other hand if p and q have a common factor M , then in view of (45) one gets the solution,

$$p\phi_1 - q\phi_2 = \phi + 2\pi k, \quad k = 0, 1, \dots, M-1. \quad (47)$$

Note that the expressions given in (41) for the amplitudes and in (46) and (47) for the phases, in the case when p and q are coprime and not coprime respectively, when substituted in the equations for the Lissajous orbit (9), in the $N \gg 1$ limit, (and for $\phi_2 = 0$), agree [31] with the solutions (18), (19), (20) guessed by Chen and coworkers [16, 19], based on their numerical study of the coordinate space probability densities associated with the coherent state (27). Besides as noted by these authors these solutions also agree with the experimental results [16]. Hence this agreement provides an *a posteriori* justification for our prescription (35) for effecting the classical limit in this problem.

4. Conclusions

In this paper we have exploited the canonical transformation (given by Louck *et al* [8]) from the commensurate anisotropic oscillator to the isotropic oscillator in order to construct appropriate $SU(2)$ coherent states for the commensurate anisotropic oscillator over a degenerate eigenspace. We have demonstrated the classical limit via the expectation values of the underlying generators. We have derived explicit expressions for the parameters in the Lissajous orbit equations in terms of the parameters of the $SU(2)$ coherent state. In particular our work completely accounts for the fact that the $SU(2)$ coherent state in general corresponds to an ensemble of Lissajous orbits.

It will be interesting to extend the procedure employed in the present paper to the case of commensurate two-dimensional anisotropic oscillator in the presence of a weak

nonlinear coupling [21] and the three-dimensional commensurate anisotropic oscillator [32] both of which have been experimentally investigated using analog optical systems recently. We hope to address these questions in our future work.

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